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On the Dirac equation with anomalous magnetic moment term and a plane electromagnetic field

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Abstract. We solve the non-minimally coupled Dirac equation for a particle with an anomalous magnetic moment in the presence of a plane electromagnetic wave, directly and from first principles. The exact wavefunction, obtained in this way, is shown to reduce to the Volkov state when the particle's anomaly is set equal to zero. Other limiting cases are also considered and their importance is pointed out.

1. Introduction

The exact solution of the Dirac equation for an electron in the field of a plane electromagnetic wave was carried out by Volkov [1] a long time ago. In modern textbooks this is done by solving the minimally coupled second-order Dirac equation [2, 3] in the presence of an external plane electromagnetic field. The state of the electron obtained in this way, known as a Volkov state, has been used extensively in exploring numerous quantum phenomena such as bremsstrahlung and photoionization [4]. In particular, in fusion research, where intense laser beams are used, the relativistic conditions in which the atoms are quickly ionized make a relativistic treatment necessary. In these conditions, coupling to the plane electromagnetic field of the laser can be extremely strong, and one is tempted to go beyond the minimal coupling scheme.

In this paper, we obtain an exact solution of the Dirac equation for a particle possessing an anomalous magnetic moment in addition to its change. The coupling in this generalized case is non-minimal and leads to a modified state for the unbound particle, which reduces to the Volkov state when the particle's anomaly is set equal to zero.

In 1968, Chakrabarti [5] constructed the solution to this problem by working out a *dynamical representation* of the Poincaré algebra of the system suitably generalized from the corresponding algebra of the free particle case. The solution thus obtained has been studied extensively and used in numerous applications.

More recent work related to the problem at hand has been published by Brown and Kowalski [6] in 1984. In 1991, Alan and Barut [7] advanced a solution to the same problem working with the first-order Dirac equation in a Weyl representation.

In this paper, we obtain the desired solution in the standard representation and from first principles, guided by the derivation in the textbooks [2, 3] of the Volkov state.

The paper is organized as follows. In section 2, we construct the non-minimally coupled second-order Dirac equation from the first-order equation and solve it along the same lines as is usually done for the case when only minimal coupling is employed. In section 3 we consider a few special cases, namely the case of pure charge and no anomalous magnetic moment and show that this one is nothing but the Volkov state, and then we consider the

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case of a neutral Dirac particle like a neutron or neutrino. We end section 3 by considering the weak-field approximation.

2. The equation and its solution

Throughout this paper, Lorentz–Heaviside units, whereby $\hbar = c = 1$, and the metric $g^{\mu\nu} = (1, -1, -1, -1)$ will be used. We employ Feynman slash notation $\not{a} \equiv \gamma^0 a_0 - \boldsymbol{\gamma} \cdot \boldsymbol{a}$ where the γ^μ are the familiar Dirac gamma matrices. The scalar product of four-vectors will be denoted by a dot, as in $a \cdot b = a^\mu b_\mu$.

Dirac's first-order equation for a particle of mass m and charge e in the presence of a plane electromagnetic field, in the minimal coupling scheme, reads

$$[(\mathrm{i}\not{\partial} - e\not{A}) - m]\psi = 0 \tag{1}$$

where the ∂_μ denote the components of the four-gradient and $A(\xi) = (0, \boldsymbol{A})$ is the external electromagnetic four-vector potential assumed to be a function of $\xi = k \cdot x$ only and satisfying the transversality condition (Lorentz gauge) $k \cdot A = 0$. Here $k = (k_0, \boldsymbol{k})$ is the plane wave propagation four-vector and $x = (t, \boldsymbol{r})$ is the coordinate four-vector of the particle.

By operating on (1) with $[(\mathrm{i}\not{\partial} - e\not{A}) - m]$, a second-order equation can be obtained whose plane wave solution is known as the Volkov state [3].

$$\psi^{(\mathrm{V})}(x) = \left(1 + \frac{e}{2k \cdot p}\not{k}\not{A}\right)\frac{u}{\sqrt{2p_0}}\mathrm{e}^{-\mathrm{i}(p \cdot x - S)} \tag{2}$$

where u is a free particle bispinor satisfying the normalization condition

$$\bar{u}u = 2m \qquad (\bar{u} = u^\dagger \gamma^0) \tag{3}$$

and

$$S = -\int_{-\infty}^{k \cdot x}\left[\frac{ep \cdot A(\xi)}{k \cdot p} - \frac{e^2 A^2(\xi)}{2k \cdot p}\right]\mathrm{d}\xi. \tag{4}$$

In (4), and everywhere in the remainder of this paper, $p = (p_0, \boldsymbol{p})$ is the (constant) energy–momentum four-vector of the particle.

Our starting point for treating a Dirac particle with an anomalous magnetic moment in the presence of a plane wave field is the *non-minimally coupled first-order equation* [2]

$$[(\mathrm{i}\not{\partial} - e\not{A}) - m + a\sigma^{\mu\nu}F_{\mu\nu}]\psi = 0. \tag{5}$$

In this equation $a = \frac{1}{2}\kappa e/2m$, where κ is the particle's anomaly (for an electron $\kappa = \frac{1}{2}(g-2) = \alpha/2\pi$), $\sigma^{\mu\nu} = \frac{1}{2}\mathrm{i}[\gamma^\mu, \gamma^\nu]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Next we obtain the corresponding second-order equation by operating on (5) with $[(\mathrm{i}\not{\partial} - e\not{A}) + m]$. The result of doing so is

$$\{[(\mathrm{i}\not{\partial} - e\not{A})^2 - m^2] + a[(\mathrm{i}\not{\partial} - e\not{A}) + m]\sigma^{\mu\nu}F_{\mu\nu}\}\psi(x) = 0. \tag{6}$$

We now simplify the various terms in (6) using the fact that $k^2 = 0$. Some of the terms in equation (6) take the following forms, after some algebra

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = k_\mu A'_\nu - k_\nu A'_\mu \tag{7}$$

where primes in (7), and throughout the remainder of this paper, denote differentiation with respect to $\xi = k \cdot x$. Noting also that $k \cdot A' = \mathrm{d}(k \cdot A)/\mathrm{d}\xi = 0$, it can be shown easily that

$$\sigma^{\mu\nu} F_{\mu\nu} = 2\mathrm{i} \not{k} \not{A}'. \tag{8}$$

Furthermore [2]

$$(\mathrm{i}\not{\partial} - e\not{A})^2 - m^2 = -\partial^2 - 2\mathrm{i}e(A \cdot \partial) + e^2 A^2 - m^2 - \mathrm{i}e\not{k}\not{A}' \tag{9}$$

where $\partial^2 = \partial^\mu \partial_\mu$. The remaining terms reduce to

$$\mathrm{i}\not{\partial}\sigma^{\mu\nu} F_{\mu\nu} = -4\not{A}'(k \cdot \partial) + 4\not{k}(A' \cdot \partial) - 2\not{k}\not{A}'\not{\partial} \tag{10}$$

and

$$-e\not{A}\sigma^{\mu\nu} F_{\mu\nu} = 2\mathrm{i}e\not{k}\not{A}\not{A}'. \tag{11}$$

Putting (9)–(11) back into (6) we arrive at

$$\begin{aligned} 0 = \{&-\partial^2 - 2\mathrm{i}e(A \cdot \partial) + e^2 A^2 - m^2 - \mathrm{i}e\not{k}\not{A}' \\ &+ a[-4\not{A}'(k \cdot \partial) + 4\not{k}(A' \cdot \partial) - 2\not{k}\not{A}'\not{\partial} + 2\mathrm{i}e\not{k}\not{A}\not{A}' + 2\mathrm{i}m\not{k}\not{A}']\}\psi(x). \end{aligned} \tag{12}$$

This is the desired second-order equation we set out to find. We look for a solution of equation (12) of the plane wave form

$$\psi(x) = \mathrm{e}^{-\mathrm{i}p \cdot x} F(\xi). \tag{13}$$

When (13) is substituted into (12), a simple first-order differential equation for $F(\xi)$ is obtained which we solve immediately. To this end, we work out, in what follows, some of the terms of this equation:

$$\partial_\mu \psi = (-\mathrm{i}p_\mu F + k_\mu F')\mathrm{e}^{-\mathrm{i}p \cdot x} \tag{14}$$

$$\partial^2 \psi = \partial^\mu(\partial_\mu \psi) = [-m^2 F - 2\mathrm{i}(k \cdot p)F']\mathrm{e}^{-\mathrm{i}p \cdot x} \tag{15}$$

$$(A \cdot \partial)\psi = -\mathrm{i}(p \cdot A) F \mathrm{e}^{-\mathrm{i}p \cdot x} \tag{16}$$

$$(k \cdot \partial)\psi = -\mathrm{i}(k \cdot p) F \mathrm{e}^{-\mathrm{i}p \cdot x} \tag{17}$$

$$(A' \cdot \partial)\psi = -\mathrm{i}(p \cdot A') F \mathrm{e}^{-\mathrm{i}p \cdot x} \tag{18}$$

$$\not{\partial}\psi = (-\mathrm{i}\not{p} F + \not{k} F')\mathrm{e}^{-\mathrm{i}p \cdot x}. \tag{19}$$

When equations (13)–(19) are used in (12),

$$2\mathrm{i}(k \cdot p)F' = \{2e(p \cdot A) - e^2 A^2 + \mathrm{i}[e\not{k}\not{A}' - 4a(k \cdot p)\not{A}' + 2a\not{k}\not{p}\not{A}' - 2ea\not{k}\not{A}\not{A}' - 2ma\not{k}\not{A}']\}F. \tag{20}$$

This is a simple first-order differential equation for $F(\xi)$. Formal integration of (20) is straightforward and yields

$$F(k \cdot x) = \mathrm{e}^{-(\alpha\not{k} + \beta\not{A} + \nu\not{k}\not{A} + \delta\not{k}\not{A}\not{p})} F(-\infty) \tag{21}$$

where

$$\alpha = \frac{\kappa}{2}\frac{e}{2m}\left[\frac{eA^2}{k\cdot p} - \frac{2p\cdot A}{k\cdot p}\right] \tag{22}$$

$$\beta = \kappa\frac{e}{2m} \tag{23}$$

$$\nu = \frac{1}{2k\cdot p}\left(\frac{\kappa}{2} - e\right) \tag{24}$$

$$\delta = \frac{\kappa}{2}\frac{e}{2m}\frac{1}{k\cdot p} \tag{25}$$

and S is as given in equation (4). $F(-\infty)$ is a suitably normalized bispinor assuming that the external plane wave field is turned on slowly in the distant past.

Finally, the wavefunction of a Dirac particle with four-momentum p, charge e and an anomalous magnetic moment, in the presence of an external plane wave electromagnetic field, becomes

$$\psi_p(x) = \mathrm{e}^{-(\alpha\not{k}+\beta\not{A}+\nu\not{k}\not{A}+\delta\not{k}\not{A}\not{p})}F(-\infty)\mathrm{e}^{-\mathrm{i}(p\cdot x-S)}. \tag{26}$$

$\psi_p(x)$ as given by (26) is exact, since no approximation has been used in its derivation. It is also general, in the sense that it applies no matter how strong the external field is. Some limiting cases will be considered in the following section.

3. Special cases

Handling the interaction of a particle with an external electromagnetic field within the context of the minimal coupling scheme ignores altogether the fact that the particle possesses an anomalous magnetic moment. In other words, the Volkov state of a particle does not take into account the small, but maybe important, contribution to the particle's dynamics coming from its interaction with the field through this part of its total magnetic moment. For our general exact solution (26) of the non-minimally coupled Dirac equation to be correct, it should necessarily yield the Volkov state when κ, the particle's anomaly, is set equal to zero. When this is done, (22)–(25) give $\alpha = \beta = \delta = 0$ and $\nu = -e/(2k\cdot p)$. With this at hand, equation (26) reduces to

$$\begin{aligned}\psi_p(x) &= \mathrm{e}^{-\nu\not{k}\not{A}}F(-\infty)\mathrm{e}^{-\mathrm{i}(p\cdot x-S)} \\ &= \left(1 + \frac{e}{2k\cdot p}\not{k}\not{A}\right)F(-\infty)\mathrm{e}^{-\mathrm{i}(p\cdot x-S)}\end{aligned} \tag{27}$$

which is the Volkov state [3], provided $F(-\infty)$ is identified with the normalized free bispinor $u/\sqrt{2p_0}$. In (27), use has been made of the identity

$$(\not{k}\not{A})^n = 0 = (\not{k})^n \qquad \text{for } n \geqslant 2. \tag{28}$$

For a neutral spin-$\frac{1}{2}$ particle, like the neutrino or neutron, which interacts with an external electromagnetic field only via an anomalous magnetic moment, the solution can be written down by setting $e = 0$ in (22)–(25) and using the resulting values of α, β, ν and δ in (26). The result is

$$\psi_p(x)(\text{neutral particles}) = \left(1 - \frac{\kappa}{4k \cdot p} \not{k} \not{A}\right) F(-\infty) \mathrm{e}^{-\mathrm{i}(p \cdot x - S)}. \tag{29}$$

Finally, we take up the weak-field approximation (WFA). This is the case when the intensity of the laser field is small such that the resulting quiver energy E_q of the particle, defined as its average classical energy in an oscillating electric field, is comparable to its rest energy E_0. Retaining only terms of order one in the expansion of the first exponential in (26) leaves one with

$$\psi_p(x)(\text{WFA}) \approx \{1 - (\alpha \not{k} + \beta \not{A} + \nu \not{k} \not{A} + \delta \not{k} \not{A} \not{p})\} F(-\infty) \mathrm{e}^{-\mathrm{i}(p \cdot x - S)}. \tag{30}$$

4. Conclusion

We have solved the Dirac equation exactly for a particle with an anomalous magnetic moment interacting with a plane electromagnetic field. The resulting wavefunction has been shown to reduce to the Volkov state when the particle's anomaly is set equal to zero. Two more special cases have been considered. For a neutral particle, a wavefunction has been found which may be of some interest to work on the solar neutrino problem. A neutral particle interacts with radiation only through its anomalous magnetic moment. Our final special case is that of a weak field. This particular case may be of importance for describing the electron in its bound state [8], before photoionization takes place, as the field strength builds up, or for fields weak enough to cause deformations in the atom and no ionization at all.

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